ECED 3300
Tutorial 6

Problem 1

Determine the electrostatic potential V at a distance b away from the origin in the xy-plane due to a finite line of length l charged with the density ρ_l. The line charge lies along the $z-$axis and extends from $z = -l/2$ to $z = l/2$.

Solution

The superposition principle yields

$$V = \int_L \frac{dl\, \rho_l}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|},$$

where $\mathbf{r}$ is the position of the observation point and $\mathbf{r}'$ is the elementary charge location. It follows from the geometry of the problem that

$$\mathbf{r} = b\mathbf{a}_\rho, \qquad \mathbf{r}' = z\mathbf{a}_z.$$

Thus,

$$V = \int_{-l/2}^{l/2} \frac{dz \rho_l}{4\pi\epsilon_0 |b\mathbf{a}_\rho - z\mathbf{a}_z|},$$

or simplifying,

$$V = \frac{\rho_l}{4\pi\epsilon_0} \int_{-l/2}^{l/2} \frac{dz}{\sqrt{b^2 + z^2}}.$$

Using the table integral,

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + const,$$

and the Newton-Leibnitz formula, we arrive at the final answer

$$V = \frac{\rho_l}{4\pi\epsilon_0} \ln\left(\frac{l/2 + \sqrt{b^2 + l^2/4}}{-l/2 + \sqrt{b^2 + l^2/4}}\right).$$

Problem 2

A spherical conducting shell of radius a, centered at the origin, has a potential field

$$V = \begin{cases} V_0, & r \le a, \\ V_0 a/r & r > a \end{cases}$$

with the zero reference at infinity. Find the expression for the stored energy in this field.

Solution

Consider the relation between the field and potential

$$\mathbf{E} = -\nabla V.$$

In cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi.$$

It then follows that

$$\mathbf{E} = -\nabla V = \begin{cases} 0 & r \leq a \\ \frac{V_0 a}{r^2}\mathbf{a}_r & r > a \end{cases}$$

Hence, the energy is given by

$$W_E = \tfrac{1}{2}\int dv \epsilon_0 E^2 = \tfrac{\epsilon_0}{2}\underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta}_{4\pi} \int_a^\infty dr\, r^2 \left(\tfrac{V_0 a}{r^2}\right)^2 = 2\pi\epsilon_0 V_0^2 a.$$

Problem 3

A conducting interface $z = 0$ separates a perfect conductor in the lower half-space, $z < 0$ and a dielectric medium with permittivity ϵ, filling the upper half-space, $z > 0$. The electrostatic potential distribution in the dielectric is given in spherical coordinates, $V(r, \theta, \phi) = \frac{V_0 r \cos\theta}{\sqrt{r^2+a^2}}$, where V_0 and a are positive constants. Determine:

(a) The electric field $\mathbf{E}$ everywhere;

(b) The surface charge density induced on the interface.

Solution

(a) The field in the dielectric can be determined directly from the potential,

$$\mathbf{E} = -\nabla V. \tag{1}$$

It turns out to be convenient to transform the potential to cylindrical coordinates,

$$V = \frac{V_0 z}{\sqrt{\rho^2 + z^2 + a^2}}. \tag{2}$$

It then follows from Eqs. (2) and (1) that

$$\mathbf{E} = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{V_0}{(\rho^2 + z^2 + a^2)^{3/2}}\left[z\rho\mathbf{a}_\rho - (\rho^2 + a^2)\mathbf{a}_z\right].$$

There is, of course, no field inside the conductor. Thus,

$$\mathbf{E} = \begin{cases} \frac{V_0}{(\rho^2+z^2+a^2)^{3/2}} \left[z\rho \mathbf{a}_\rho - (\rho^2 + a^2)\mathbf{a}_z \right], & z > 0 \\ 0 & z < 0. \end{cases}$$

(b) Using the boundary conditions at the dielectric-conductor interface,

$$D_z(\rho, z = 0) = \epsilon_0 E_z(\rho, z = 0) = -\rho_s,$$

we obtain the sought surface charge density

$$\rho_s = \frac{\epsilon_0 V_0}{\sqrt{\rho^2 + a^2}}.$$

Problem 4

Two homogeneous dielectric regions $z \leq 0$ (region 1) and $z \geq 0$ (region 2) have dielectric constants $\epsilon_{r1} = 4$ and $\epsilon_{r2} = 2$, respectively. Given the electric field $\mathbf{E}$ in region 1, $\mathbf{E}_1 = \mathbf{a}_x + 3\mathbf{a}_y + 5\mathbf{a}_z$, find the electric field $\mathbf{E}_2$ and the electric flux density $\mathbf{D}_2$ in region 2. You may leave your results in terms of ϵ_0.

Solution

The unit normal to the interface $z = 0$ between the two media is $\mathbf{a}_z$. Hence, the normal and tangential components of the electric field in region 1 are

$$\mathbf{E}_{1n} = (\mathbf{E} \cdot \mathbf{a}_z)\mathbf{a}_z = 5\mathbf{a}_z,$$

$$\mathbf{E}_{1t} = \mathbf{E} - (\mathbf{E} \cdot \mathbf{a}_z)\mathbf{a}_z = \mathbf{a}_x + 3\mathbf{a}_y.$$

The corresponding components of the electric field in region 2 are

$$\mathbf{E}_{2n} = \alpha \mathbf{a}_z,$$

$$\mathbf{E}_{2t} = \mathbf{a}_x + 3\mathbf{a}_y.$$

Here α is an unknown constant, and the tangential component is the same due to the boundary condition: $\mathbf{E}_{1t} = \mathbf{E}_{2t}$. The second boundary condition, $\mathbf{D}_{1n} = \mathbf{D}_{2n}$ implies

$$5\epsilon_{r1} = \alpha \epsilon_{r2}, \Longrightarrow \alpha = 10.$$

Thus,

$$\mathbf{E}_2 = \mathbf{E}_{2n} + \mathbf{E}_{2t} = \mathbf{a}_x + 3\mathbf{a}_y + 10\mathbf{a}_z,$$

and

$$\mathbf{D}_2 = \epsilon_0 \epsilon_{r2} \mathbf{E}_2 = 2\epsilon_0 \mathbf{a}_x + 6\epsilon_0 \mathbf{a}_y + 20\epsilon_0 \mathbf{a}_z.$$